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Rapports de Recherche

N° 581

ROBUST STABILITY OF EXPLICIT ADAPTIVE CONTROL WITHOUT PERSISTENT EXCITATION

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Novembre 1986

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ROBUST STABILITY OF EXPLICIT ADAPTIVE CONTROL

WITHOUT PERSISTENT EXCITATION

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PUBLICATION INTERNE N° 317 - Octobre 86

60 pages

Abstract.- This paper deals with adaptive control, based on explicit identification. The problem of the identified model stabilizability is solved in the passive approach, i.e. without requiring persistingly exciting inputs. This solution is robust, covering time varying processes, unstructured model errors and underestimated model order.

Key words.- Adaptive control, Robustness, Indirect, Passive

STABILITE ROBUSTE DE LA COMMANDE ADAPTATIVE EXPLICITE SANS EXCITATION PERSISTANTE

Résumé.- Ce papier concerne la commande adaptative basée sur une identification explicite. Le problème de la stabilisabilité du modèle identifié est résolu dans l'approche passive, c'est à dire sans nécessiter des entrées continuellement excitantes. La solution est robuste, dans le contexte de processus variable dans le temps, avec erreurs de modèle non structurées et ordre sous estimé.

Mots clés : Commande adaptative, Robustesse, Indirecte, Passive·

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INTRODUCTION

According to the certainly equivalence principle [1], the most natural approach for adaptive control associates some real time identification method, and continuous updating the control law from the identified model.

Before 1980, almost everybody thought that this approach could not lead to any complete theoretical analysis, in a somewhat general framework.

That is the reason why, between 1970 and 1980 most of the theoretical works concerned the "direct approach", i.e. direct adjustment without explicit identification (see e.g. [2] to [6]), in spite of major limitations, such as minimum phase hypothesis.

In the early eighties it was recognized that the stability conditions in indirect approach could be formulated in a very general and fruitful form [7,8]. Typically, these conditions are the following :

i) the identified model must satisfy some smallness conditions concerning :

- the equation error $\nu(t)$*, which appears in the process equations, when written with the identified* $\hat{\theta}(t)$*, and the observed input output signals,*

- the evolution rate of $\hat{\theta}(t)$*.*

ii) the identified model must be stabilizable, for the adjustment algorithm may lead to a control parameter vector $k(\hat{\theta})$*, such that the closed loop characteristict polynomial involved by* $\hat{\theta}(t)$ *and* $k(\hat{\theta})$ *be strictly Hurwitz.*

These conditions do not restrict the choice of control algorithm. Then the adaptive control problem begins to be widely recognized as a pure identification problem.

In fact, most of the classical identification methods, based on the Prediction Error Method [9], directly satisfy the first above condition (i).

Unfortunately these methods do not present any special properties concerning the above stabilizability condition (ii). In order to solve this key problem, the so called active approach states that the process must be persistingly excited, either by means of the reference input, either by additive extra signals. Then, the convergence of $\hat{\theta}(t)$ towards a vicinity of the exact (and stabilizable) parameters $\theta^*(t)$ solves the problem [10].

However, artificial exciting inputs are generally nocuous for the control purpose itself, especially in the regulation mode.

An other solution lies in some projection techniques of $\hat{\theta}(t)$ in an a priori given stabilizable (or admissible) domain D_A. This approach holds only if D_A is simply connected and convex, which is a very restrictive assumption.

A first general solution to the stabilizability problem was proposed by de Larminat [11]. It is based on the use of two models :

- a model $\hat{\theta}(t)$, delivered by an ordinary least square algorithm,
- a model $\bar{\theta}(t)$, constrained to be stabilizable, and then usable for control.

When $\hat{\theta}(t)$ escapes from the admissible domain D_A, it is reinitialized into the intersection of D_A with an ellipsoid centered on $\hat{\theta}(t)$, defined by $\|\bar{\theta} - \hat{\theta}\|_{P^{-1}} \leqslant 1$, where $P(t)$ is the classical L.S. matrix.

Althouth this algorithm was designed having in mind the noisy case, the complete proof of satisfying (i) and (ii) was performed in [11] only in the noise-free case, and one could fear it did not present any robustness property with respect to noise. The same criticism could be objected to a similar work of Lozano and Goodwin [12].

In [13], de Larminat presented a solution based on the same principles, including a robustness analysis, with regard not only to noise, but also to unstructured model errors (non linearities, underestimated model order), and even to time varying processes.

The present work consists in a reviewed and completed version of [13], then, it offers a very complete solution to the indirect passive approach in adaptive control. This solution takes form as a typical example of an identification algorithm, satisfying the above conditions (i) and (ii). In addition, a theorem of robust stability for time varying systems plays the role of a technical lemma, for the stability analysis of adaptive control systems. This theorem is widely derived from a previous work of Praly [14].

For simplicity, only one identification algorithm is presented. Though a general class is not explicitely given, its basic features are the following :

- the algorithm belongs to the Least-Square type, involving a significant matrix P, which is necessary for the stabilizability correction below. It means that scalar gains must be discarded,

- the classical forgetting is introduced, for time varying process identification,

- the prediction error is normalized by the norm of the observation vector,

- an attraction towards zero permits to keep $\hat{\theta}$ and P bounded,

- all the previous characteristics are organized so that the pair $(\hat{\theta}, P)$ define a certainity area around $\hat{\theta}$,

- from this first algorithm, a stabilizable model $\tilde{\theta}(t)$ is deduced : the pair $(\hat{\theta},P)$ is used as a permanent memory of the past input output data, in order to define a possible reinitialization for $\hat{\theta}(t)$, when it escapes the admissible domain D_A.

The general principles above are quite reasonable and their application is relatively simple. However, it was not straightforward that they could lead to a proof of overall stability for adaptive control systems. For simplicity, this paper deals with the continuous time case, for which complete proofs are provided.

In section 2, the hypotheses and notations are introduced, for the control of monovariable, continuous time, n-order processes. The concept of admissible domain D_A is also defined. Section 3 is devoted to a theorem of robust stability, particularly suitable for adaptive control analysis. This theorem formalizes that a process $\{\dot{x} = F(t)x(t)+w(t)\}$ is stable if there exist convenient upper bounds for $\|w(t)\|$ and $\|\dot{F}(t)\|$. In section 4, the primary identification algorithm is described and it is shown (section 5) that the robustness theorem applies, assuming that $\hat{\theta}(t)$ remains uniformly stabilizable.

The heart of the paper relies in section 6, where the admissible model $\bar{\theta}(t)$ is introduced : It is shown that $\bar{\theta}(t)$ simulaneously satisfies the required conditions (i) and (ii). The key problem was to exhibit an convenient upperbound for the number of possible reinitialisation of $\bar{\theta}(t)$, in a given interval of time. The main conclusions are provided in section 7. The appendix describes the proof of the robust stability theorem.

2. PROCESS AND CONTROL LAW : NOTATIONS. EQUATIONS AND HYPOTHESES

First consider a deterministic, time invariant process, input u(t), output y(t), which is equivalently described by the following equations (2-1) to (2.3) :

$$\frac{d^n u}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + \dots + a_{n-1}\frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1}u}{dt^{n-1}} + \dots + b_{n-1}\frac{du}{dt} + b_n u \qquad (2.1)$$

$$A(s)y = B(s)u \qquad (2.2)$$

$$z(t) = \phi(t)^T\theta \qquad (2.3)$$

where :

s is the differenciation operator

$$A(s) \triangleq s^n + a_1 s^{n-1} + \ldots + a_n \tag{2.4}$$

$$B(s) \triangleq b_1 s^{n-1} + \ldots + b_n \tag{2.5}$$

$$z(t) \triangleq \frac{d^n y}{dt^n} \tag{2.6}$$

$$\phi^T(t) \triangleq \left[\frac{d^{n-1}y}{dt^{n-1}} \ldots \frac{dy}{dt} \quad y \;\middle|\; \frac{d^{n-1}u}{dt^{n-1}} \ldots \frac{du}{dt} \; u \right] \tag{2.7}$$

$$\theta^T \triangleq \left[-a_1 \ldots -a_{n-1} \;\; -a_n \;\middle|\; b_1 \quad b_{n-1} \quad b_n \right] \tag{2.8}$$

A time-invariant control law is defined :

$$\frac{d^n u}{dt^n} = k^T \phi(t) + k_r y_r(t) \tag{2.9}$$

where : y_r is the reference signal,

k_r a scalar gain,

$$k^T \triangleq - \left[q_1 \ldots q_n \;\middle|\; p_1 \ldots p_n \right] \tag{2.10}$$

Then, the control law can be written as

$$P(s)\, u = - Q(s) y + k_r y_r \tag{2.11}$$

Nota : The feedback transfer Q(s)/P(s) is strictly proper (i.e. $q_0 = 0$ in Q(s)), which yields more simplicity without appreciable loss of generality. Similarly, one can introduce a polynomial $K_r(s)$ instead of $k_r(s)$. Moreover, the orders of A, B, C, D could be lower than n.

The closed loop equations are written :

*either into the polynomial form :

$$(AP + BQ)\ y = B\ k_r\ y_r \tag{2.12}$$

$$(AP + BQ)\ u = A\ k_r\ y_r \tag{2.13}$$

*either into the state form :

$$\dot{\phi} = F\phi + w \tag{2.14}$$

where :

$$F \triangleq \left[\begin{array}{ccc|c|c} & & & \theta^T & \\ \hline 1 & & & & \\ & \ddots & & 0 & 0 \\ & & 1 & & \\ \hline & & & k^T & \\ \hline & & 1 & & \\ 0 & & & \ddots & \\ & & & 1 & 0 \end{array}\right] \quad , \quad w \triangleq \left[\begin{array}{c} \vdots \\ 0 \\ \vdots \\ \hline k_r\ y_r \\ \hline \vdots \\ 0 \\ \vdots \end{array}\right] \tag{2.15}$$

In order to satisfy the required performances, k and k_r are adjusted to θ, according to an application $\mathscr{A}$:

$$\begin{array}{ccc} R^{2n} & \xrightarrow{\mathscr{A}} & R^{2n+1} \\ \theta & \longmapsto & \left[\begin{array}{c} k \\ \hline k_r \end{array}\right] \end{array}$$

This adjustment law $\mathscr{A}$ must at least satisfy :

- $\left[A(s)\ P(s) + B(s)\ Q(s)\right]$ be an Hurwitz polynomial (2.16)

(for stability)

- $a_n p_n + b_n q_n = b_n k_r$ (2.17)

(for zero static error)

All the classical control methods for linear systems lead to adjustment laws which satisfy at least (2.16).

However, according to the control method, then it exists some constraints on θ, for example :

- A(s) and B(s) must be coprime, when using all-pole placement,

- In addition, B(s) must be Hurwitz (minimum phase condition) for pole and zero placement, or perfect model following, or minimum output variance,

- For any method : if A(s) and B(s) are not coprime, their common factor must be Hurwitz. This necessary stabilizability condition is also sufficient for various methods, such as Linear Quadratic optimization,

- In order to satisfy (2.17), b_n must be non zero.

It follows that, for a given control method, θ must belong to a specified subset of the parametric space, i.e. an Admissible Domain : $D_A \subset R^{2n}$. For many adjustment laws $\mathscr{A}$, D_A do not reduce to the stabilizable domain, but is strictly included in it.

In order to deal with concrete adaptive control problems, introduce a set of properties, defining the admissible domains.

Let $\mathcal{A}$ an adjustment law, which yields $k = k(\theta)$, and thus $F = F(\theta)$ (from 2.15). A subspace D_A is said to be admissible *with respect to $\mathcal{A}$ if there exists some positive constants Ω_F, ω_F, M_R and M_D such that, for any $\theta \in D_A$, the following P_F, P_R and P_D properties hold :*

$$P_F : \begin{cases} \|F(\theta\| \leqslant \Omega_F & (2.18) \\ R_e \quad (\lambda_i(\theta)) \leqslant -\omega_F < 0 & (2.19) \end{cases}$$

(λ_1 λ_2 ... λ_{2n} : Eigenvalues of $F(\theta)$)

$$P_r : \quad |k_r| \leqslant M_R \tag{2.20}$$

$$P_D : \quad \forall\ \theta_1, \theta_2 \in D_A : \frac{\| F(\theta_1)-F(\theta_2) \|}{\| \theta_1 - \theta_2 \|} \leqslant M_D \tag{2.21}$$

Comments :

* (2.18) is a simple boundedness condition
* (2.19) involves the asymptotic stability of F.

It could be added more restrictive conditions (pole damping or others) but (2.19) is the strict minimum to be required.

* (2.20) will be associated with a boundedness hypothesis on y_r.
* Finally, the continuity condition P_D will be necessary when analysing adaptive control systems, where θ (or $\hat{\theta}$) becomes time-varying.

A basic example : pole placement control

An arbitrary Hurwitz polynomial is given :

$$D(s) \triangleq s^{2n} + d_1 s^{2n-1} + \dots + d_{2n-1}s + d_{2n}$$

The roots $\sigma_i (i=1, \dots 2n)$ are assumed to satisfy :

$$R_e(\sigma_i) \leqslant -\omega_F \qquad (2.22)$$

and $$d_{2n} \neq 0$$

Define $k(\theta)$ as the solution of the diophantian equation :

$$A(s)\, P(s) + B(s)\, Q(s) = D(s)$$

which is equivalent to the linear system

$$S(\theta)\, k = d$$

where

$$S(\theta) \triangleq \left[\begin{array}{cccc|cccc} 1 & & & & 0 & & & \\ a_1 & \ddots & & & b_1 & \ddots & & \\ \vdots & \ddots & \ddots & & \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 & \vdots & & \ddots & 0 \\ \hline a_n & \ddots & & a_1 & b_n & \ddots & & b_1 \\ & \ddots & & \vdots & & \ddots & & \vdots \\ & & & a_n & & & & b_n \end{array}\right], \quad k \triangleq \left[\begin{array}{c} p_1 \\ \vdots \\ p_n \\ \hline q_1 \\ \vdots \\ q_n \end{array}\right], \quad d \triangleq \begin{bmatrix} d_1 \\ \vdots \\ \vdots \\ d_{2n} \end{bmatrix} \qquad (2.23)$$

Then $\mathcal{R}$ is defined :

$$\mathcal{R} : \left[\begin{array}{ll} k(\theta) = S^{-1}(\theta) d & (2.24) \\ & \\ k_r(\theta) = \dfrac{a_n p_n + b_n q_n}{b_n} = \dfrac{d_{2n}}{b_n} & (2.25) \end{array}\right.$$

Now, it is proposed to define D_A as follows

$$\boxed{\theta \in D_A \iff \begin{cases} \|\theta\| \leqslant R_\theta & (2.26) \\ \|k(\theta)\| \leqslant R_k & (2.27) \\ \|k_r(\theta)\| \leqslant M_r & (2.28) \end{cases}}$$

where R_θ, R_k, M_r are a priori given constants.

It is clear that D_A is an admissible domain. In effect, for any $\theta \in D_A$:

* (2.26) and (2.27) imply the existence of a bound Ω_F in (2.18)

* (2.22) implies (2.19)

* from (2.24) and (2.25), it follows that $k(\theta)$ components are some rational fractions of θ, which denominators are not zero for any θ satisfying (2.26) and (2.27). Thus their derivatives are bounded for any $\theta \in D_A$. M_D exists in (2.21).

N.B.: In (2.18) to (2.21), only the existence of the bounds Ω_F, ω_F, M_r, M_D is required. In the example above,

* M_r is directly given

* Ω_F can be deduced from the given R_θ and R_k

* M_D will be more difficult to deduce from $D(s)$, R_θ and R_k

* moreover, the exact shape of D_A is practically impossible to deduce. It is not convex, neither simply connected since the constraint (from 2.25, 2.28)

$$|b_n| \geqslant \frac{d_{2n}}{M_r} \tag{2.29}$$

at least divises it into two disconnected subspaces.

However the knowledge of the shape of D_A will not be necessary in the sequel. Similarly, the knowledge of the bounds Ω_F, ω_F, $\dot{M}_r$, M_D could be usefull in order to quantify the robustness of the given control law $\mathscr{A}$, but is not necessary for proving the existence of this robustness.

Finally, it will be necessary to introduce a strongly admissible domain D_{SA}, based on the following assumptions :

* D_{SA} is strictly included into D_A
* the distance δ between $\overline{D_A}$ and D_{SA} is non zero :

$$\delta \triangleq \underset{\substack{\theta_1 \in D_A \\ \theta_2 \in \bar{D}_A}}{\mathrm{Min}} \quad \|\theta_1 - \theta_2\| > 0 \tag{2.30}$$

In the case of pole placement, D_{SA} will be similarly defined, from some R'_θ, R'_k, M'_r :

$$R'_\theta < R_\theta \; , \; R'_k < R_k \; , \; M'_r < M_r$$

The existence (not the knowledge) of δ is straightforward, knowing that $k(\theta)$ is bounded and differenciable when $\theta \in D_A$.

From the above example, it is clear that most of the reasonable control methods could exhibit similar properties : the admissible domain will be preferably defined from some thresholds occuring in the implementation of the method. It is obviously easier to determine the admissibility of θ from the norm of $k(\theta)$, than from a direct measure (if exists !) of the coprimeness of $A(s)$ and $B(s)$.

In addition, D_A will depend also from the a priori knowledge of the considered class of process.

In some very specific problems, D_A could be defined as a local vicinity of an a priori given model. Then, the convexity hypothesis could hold, but that will not be assumed in the sequel.

3. A THEOREM OF ROBUST STABILITY FOR SLOWLY TIME VARYING SYSTEMS

Reconsider now the state equation

$$\dot{x} = F(x) + w$$

in which F and w do not necessarily exhibit the structure defined in eq. (2.15).

Now, F(t) will be a function of time, and w(t) will include possible model errors.

More precisely, it means that, x(t) beeing some function of time, F(t) being some given model, either a priori given, either an identified one, then w is now defined as the difference : $w \triangleq \dot{x} - Fx$.

It is clear now that even if $\{R_e(\lambda_i(t)) \leqslant -\omega_F\}$ for every t, the stability is no longer assumed, unless $\|\dot{F}\|$ and $\|w\|$ be "relatively small". Various upperbounds could be proposed for $\|\dot{F}\|$ and $\|w\|$. In this section, such upperbounds will be selected, for their further interest when dealing with robust stability in adaptive control.

Theorem 1

Let $x(t)$, $w(t)$, $F(t)$ be some functions of time (dim. $(n_F x1)$, $(n_F x1)$, and $(n_F x n_F)$). Assume that the following properties hold for every t, τ and $T > 0$:

$$\frac{dx}{dt} = F(t)\ x(t) + w(t) \tag{3.1}$$

$$\|x\ (0)\| \text{ is finite.} \tag{3.2}$$

There exist ω_F and Ω_F, positive constants, such that :

$$R_e \left\{ \lambda_i(t) \right\} \leqslant -\omega_F < 0 \quad (i = 1, 2, \dots n_F) \tag{3.3}$$

$$\|F(t)\| \leqslant \Omega_F < \infty \tag{3.4}$$

Assume also that :

$$\int_t^{t+T} \frac{\|\dot{F}\ (\tau)\|}{k_d}\ d\tau \leqslant T + T_1 \tag{3.5}$$

$$\int_t^{t+T} \frac{\|w(t)\|^2}{M_w + k_x \|x(t)\| + k_\xi\ \xi(t)^2}\ d\tau \leqslant T + T_2 \tag{3.6}$$

where $\xi(t)$ satisfies

$$0 \leqslant \xi(0) < \infty \tag{3.7}$$

$$T_\xi\ \dot{\xi}(t) = -\ \xi(t) + \|x(\tau)\| \tag{3.8}$$

and where

- T_1, T_2, M_w are some positive constants (possibly large)
- T_ξ, k_d, k_x, k_ξ are positive constants, which depend on ω_F and Ω_F.

Then : $x(t)$ is uniformly bounded.

For T_ξ, k_d, k_x, k_ξ, the following expressions are proposed :

$$T_\xi \leqslant T_F \tag{3.9}$$

$$k_d \leqslant \frac{\tau_F}{2T_F^{\,3}} \tag{3.10}$$

$$k_x \leqslant \frac{1}{16T_F} \sqrt{\frac{\tau_F}{T_F^{\,3}}} \tag{3.11}$$

$$k_\xi \leqslant \frac{1}{16T_F} \sqrt{\frac{T_\xi}{T_F^{\,3}}} \tag{3.12}$$

where

$$T_F \triangleq \frac{n_F^2}{2\omega_F} \left(\frac{2\Omega_F}{\omega_F}\right)^{2n_F} \tag{3.13}$$

$$\tau_F \triangleq \frac{1}{\Omega_F} \tag{3.14}$$

Comments

The above expressions are not unique, and more efficient ones could be possibly found. Obviously, they depend on certain arbitrary choice occuring in the derivation of the proofs (see appendix). In concrete application (3.10) to (3.14) show that k_d, k_x, k_ξ could often be very small. That is the price to be aid for the generality of the hypotheses (3.1) to (3.8).

Under some additional constraints (damping coefficients of the complex conjugate λ_i, for example) k_d, k_x, k_ξ could be larger. However, our present interest does not lie in the robustness problem by itself, but in its application to adaptive control.

From this point of view, we prefer to base our work upon a minimal set of properties, like (3.3) and (3.4), for which :

* the above non zero constants k_d, k_x, k_ξ exist
* the robustness involved by (3.5) - (3.6) can be quantified (at least theoretically)
* the quantification depends only on some characteristics of F(t)
* the general form of (3.9) to (3.13) permits to inlight some general connections beetwen the characteristics (ω_F, Ω_F) and the robustness.

For example, if F(t) is the closed loop matrix (2.15), and if $k(\theta)$ is a control based on some "large gain principle", then Ω_F is large. The robustness dependance can be analyzed via (3.9) to (3.14).

Consider now the feature of the upperbounds (3.5) and (3.6).

Taking T large, k_d bounds the mean speed rate of F(t). Taking $T \rightarrow 0$, T_1 bounds the magnitude of jumpsF(t). Then (3.5) permits continuous, but slow change in F(t), and also large, but rare jumps.

Now, analyze how the bound (3.6) may cover unstructured model errors : Assume an exact, n_F-order, non linear model $f^*(\cdot)$.

$$\dot{x} = f^*(x) + w_e \tag{3.15}$$

where w_e is an exogeneous bounded input :

$$\|w_e\| \leqslant M_w \tag{3.16}$$

Now, F is a given model matrix. From (3.1), w(t) is defined as

$$w \triangleq w_e + f^*(x) - F\,x \tag{3.17}$$

If $f^*(x)$ is weakly non linear, so that, for any x the following (3.18) inequality hold

$$\|f^*(x) - Fx\| \leqslant k_x \tag{3.18}$$

then (3.6) will be satisfied.

Consider now an exact model of order greater than n_F. For simplicity we reduce to the linear, deterministic case, where $x(t)$ is assumed to be the solution of an extended state, exact model :

$$\frac{d}{dt}\left[\begin{array}{c} x \\ \hline \xi \end{array}\right] = \left[\begin{array}{c|c} F & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right] \left[\begin{array}{c} x \\ \hline \xi \end{array}\right] \tag{3.19}$$

Then $\dot{x} = F\,x + w$ (3.20)

where F is assumed to be the given model and w is the output of :

$$\dot{\xi} = F_{22}\,\xi + F_{22}\,x \tag{3.21a}$$

$$w = F_{12}\,x \tag{3.21b}$$

An other form of (3.21) is given by the convolution :

$$w(t) = H(t) * x(t) \tag{3.22}$$

where $H(t)$ is the impulse response of the triplet $\{F_{22}, F_{21}, F_{12}\}$

If F_{22} is exponentially stable, F_{12} or F_{21} sufficiently small, there exist T_ξ and k_ξ, such that :

$$\|H(t)\| \leqslant k_\xi\, e^{-t/T_\xi} \tag{3.23}$$

and then $$\|x(t)\| \leqslant k_\xi \;\xi(t) \tag{3.24}$$

which satisfies (3.6) to (3.8)

More generally (3.6) to (3.8) permit to handle unstructered model errors, of the general form

$$w(t) = w(t, x(\tau) \mid \tau \leqslant t) \quad . \tag{3.25}$$

It could be thought that the simple inequality

$$\frac{||w||}{M_w + k_x ||x|| + k_\xi \xi} \leqslant 1 \tag{3.26}$$

could be sufficient for most of the robustness analysis, and (3.6) be an academic refinement.

In fact, (3.6) will be really necessary for adaptive control analysis, where F(t) results from an identified model. Then, w(t) depends on the equation error, and will satisfy an integral inequality like (3.6). Similarly, the identified $\hat{\theta}(t)$ will satisfy an inequality like (3.5).

Finally, the above theorem will reveal itself as a basic "Technical Lemma", when analysing robust stability of indirect adaptive control systems.

Proof of the theorem

The proof is detailed in appendix A. It is based on a Lyapunov type approach, using the function :

$$W(t) = x^T(t)\ \Sigma(t)\ x(t) + T_\xi\ \xi^2(t) \tag{3.27}$$

where $\Sigma(t)$ is the solution of the Lyapunov equation :

$$F^T(t)\ \Sigma(t) + \Sigma(t)\ F(t) + 2I = 0 \tag{3.28}$$

According to the stability theory, (3.3) and (3.5) imply the existence of upper and lower bounds :

$$\tau_F \; I \leqslant \Sigma(t) \leqslant T_F \; I \tag{3.29}$$

The first part of the Appendix is devoted to deriving the explicit forms (3.12) and (3.13) for T_F and τ_F.

The second part concerns the relationship between $\dot{\Sigma}$ and $\dot{F}$: it is shown that

$$\|\dot{\Sigma}\| \leqslant T_F^{\;2} \; \|\dot{F}\| \tag{3.30}$$

Then, from (3.27) to (3.30), it is proved in a third part that W(t) is bounded.

4. AN IDENTIFICATION ALGORITHM ORIENTED TOWARD ADAPTIVE CONTROL

In the passive approach, the inputs are not necessarily persistingly exciting. Thus, it is unrealistic to aim at tracking some "exact" model $\theta^*(t)$.

The only reasonable goal is to reduce the magnitude of the equation error :

$$\nu(t) \triangleq z(t) - \phi^T(t) \; \hat{\theta}(t)$$

and the speed :

$$\dot{\hat{\theta}}(t) \triangleq \frac{d\hat{\theta}(t)}{dt}$$

Both must be "relatively" small for the model to be usable.

Clearly, it will be interesting to bound $\nu(t)$ and $\dot{\hat{\theta}}(t)$ by some expressions in connection with inequalities (3.5) and (3.6), arizing in the robustness theorem.

Fortunately, classical algorithms, such as continuous-time Least-Squares algorithm, directly satisfy such inequalities.

However, L.S. must be slightly modified in order to satisfy some additionnal requirements :

- $\hat{\theta}(t)$ must be kept bounded
- the real process may be noisy, slowly time varying, weakly non linear, and underestimated order
- the algorithm must be Least-Squares type, not only gradient type. In other words, a significant "variance covariance" matrix P must exist
- upper bound and lower bound over P must exist
- the identified model must be admissible, i.e. at least stabilizable
- in order to solve the stabilizability problem (see §5), the pair $\hat{\theta}(t)$, $P(t)$ is required to define a certainly domain around $\hat{\theta}(t)$
- all that must be obtained without prejudice to inequalities over $\nu(t)$ and $\dot{\hat{\theta}}(t)$.

In this paper, we present only, as a characteristic example, an algorithm which satisfies all the required conditions. One can imagine possible variations, alghough the above conditions are not straight-forward to meet together.

Define first the process to be identified,control input $u(t)$, output $y(t)$.

Assume that there exists some vector θ*(t), so called "nominal" or "best model", and denote v(t) the equation error :

$$v(t) \triangleq z(t) - \phi^T(t)\ \theta^*(t) \tag{4.1}$$

where z(t) and φ(t) are defined above (2.6), (2.7).

Some constants σ, ρ, e, β are **given**, such that :

$$\frac{v(t)}{s(t)} \leqslant 2 \tag{4.2}$$

$$\text{where } s(t) \triangleq \sigma + r\|\ \phi(t)\| + \rho\ \psi(t) \tag{4.3}$$

ψ(t) beeing solution of

$$\psi = -\ \beta(\psi(t) - \|\phi(t)\|) \qquad (0 \leqslant \psi(0) < \infty) \tag{4.4}$$

$\|\hat{\theta}(t)\|$ is bounded by a **given** R :

$$\|\theta^*\|^2 \leqslant R^2 \tag{4.5}$$

For further simplification the bound for $\|\dot{\theta}^*\|$ is defined as follows : a positive α is given, such that :

$$\|\dot{\theta}^*\| \leqslant \alpha^2 \quad \frac{R^2 r^2}{R^2 + r^2} \tag{4.6}$$

Remarks

* From (4.2), (4.3), (4.4), the process can be non linear, and underestimated order.

* (4.5) defines an a priori, spheric domain. One can consider an ellipsoidal domain centered around any given θ_N :

$$(\theta^* - \theta_N)^T Q(\theta^* - \theta_N) \leqslant R^2 \qquad (4.7$$

Where Q is non singular, factorizable into :

$$Q = S^T S \tag{4.8}$$

(4.1) can be rewritten as :

$$v(t) = |z(t)-\phi^T\theta_N| - |\phi^T S^{-1}|\ |S(\theta^*-\theta_N)| \tag{4.9}$$

Then replace z by $(z - \phi^T\theta_N)$

ϕ^T by $\phi^T S^{-1}$

θ^* by $S(\theta^* - \theta_N)$

The new θ^* satisfies now (4.5)

* Note that the given constant r plays a role in the inequalities on $v(t)$ and on $\dot{\theta}^*(t)$ (cf. (4.5) and (4.6)). Thus for a linear, but time varying system, r could be zero from (4.5) but not from (4.6).

Then, from the given σ, r, e, β, r, α, the following identification algorithm is proposed :

$$\dot{\hat{\theta}}(t) = \alpha\ (P(t)\ \phi(t)\ \frac{\nu(t)}{s^2(t)} - \frac{P(t)\ \hat{\theta}(t)\ \lambda}{R^2} \tag{4.10}$$

$$\dot{P}(t) = \alpha(\frac{-P(t)\ \phi(t)\ \phi(t)^T P(t)}{S^2(t)} + P(t) - \frac{P(t)^2}{R^2}) \tag{4.11}$$

with $\|\hat{\theta}(0)\| \leqslant R$ (4.12)

$$P(0) = R^2 I \tag{4.13}$$

where $\nu(t) \triangleq z(t) - \phi^T(t)\ \hat{\theta}(t)$ (4.14)

$s(t)$ is defined above (4.3), (4.4) ,

and where $\lambda = 0$ if $\|\hat{\theta}\| < R$ (4.15)

$\lambda = 1$ if $\|\hat{\theta}\| \leqslant R$ (4.16)

Comments

The above algorithm could be compared with the exponentialled wheighted L.S., which are given by

$$\dot{\hat{\theta}} = P\ \phi\ \nu \tag{4.17}$$

$$\dot{P} = P\phi\ \phi^T p + \alpha P \tag{4.18}$$

The differences first concern the factor α/s^2 on $P\phi\nu$ and $p\phi\phi^T P$. In connection with the hypotheses (4.2) to (4.6), this factor will permit to derive a certainty domain from the pair $(\hat{\theta}, P)$, and also the desired inequalities bounding ν and $\dot{\theta}$.

Remarks that α can be zero in 4.18 (no forgetting) but not in (4.10), (4.11). That will be the price of the certainty domain (4.25). On the other hand, the additional terms $P\hat{\theta}\lambda/R^2$ and P^2/R^2 work in association in order to bound $\hat{\theta}$ and P.

The algorithm (4.10) to (4.16), under the hypotheses (4.1) to (4.6) exhibits the following properties.

Property 1 : P is bounded

$$\boxed{\begin{aligned} &P(t) \geqslant \frac{R^2\ r^2}{R^2 + R^2}\ I \qquad (4.19) \\ &P(t) \leqslant R^2 I \qquad (4.20) \end{aligned}}$$

Proof

Denote $M(t) \overset{\Delta}{=} P^{-1}(t)$ (4.21)

wich yield from (4.11) :

$$\dot{M} = \alpha\ \frac{\phi\phi^T}{s^2} + M - \frac{I}{R^2} \tag{4.22}$$

Let a be any constant vector, and $U(t) \triangleq a^T M a$

From (4.22) : $\dot{U} = \alpha \left(\frac{a^T\phi}{s}\right)^2 - U + \|a\|^2/R$

From (4.3) : $\left(\frac{a^T\phi}{s}\right)^2 \leqslant \frac{\|a^2\| \ \|\phi\|^2}{(\sigma + r \ \|\phi\| + \rho\psi)^2} \leqslant \frac{\|a^2\|}{r^2}$

Thus : $- U + \frac{\|a\|^2}{R^2} \leqslant \frac{\dot{U}}{\alpha} \leqslant U + \frac{\|a\|^2}{R^2} + \frac{\|a\|^2}{r^2}$

Now : $U(0) = \frac{\|a\|^2}{R^2} \in \left[\frac{\|a\|^2}{R^2} , \ (\frac{\|a\|^2}{R^2} + \frac{\|a\|^2}{r^2}\right]$

When U(t) rise the upper bound of the above interval, $\dot{U}(t)$ becomes non positive. Thus U(t) cannot cross over the bound. Idem for the lower bound.

Thus, (4.19) and (4.20) are proved, via :

$$M \geqslant I/R^2 \tag{4.23}$$

$$M \leqslant I\left(\frac{1}{R^2} + \frac{1}{r^2}\right) \tag{4.24}$$

Comments

Clearly, the term R^2/I, in (4.11), prevents the possible divergence of P, when using exponential forgetting.

On the other hand, if $\phi(t)$ is not bounded, P could be infinitely small in the ordinary L.S. (4.17, 4.18). A first advantage of the normalization involved by $r^2\|\phi\|$ in s^2 (cf. 4.3) is to yield the lower bound (4.19).

Property 2 : Existence of a non trivial certainty ellipsoid :

$$\boxed{(\hat{\theta} - \theta^*)^T P^{-1} (\hat{\theta} - \theta^*) \leqslant 16} \tag{4.25}$$

Proof

define $$\tilde{\theta} \triangleq \hat{\theta} - \theta^* \tag{4.26a}$$

$$V \triangleq \tilde{\theta}^T P^{-1} \tilde{\theta} \tag{4.26b}$$

by differenciation, using (4.18)

$$\dot{V} = \tilde{\theta}^T \dot{M} \tilde{\theta} + 2\tilde{\theta}^T P^{-1} (\dot{\hat{\theta}} - \dot{\theta}^*)$$

Then, from (4.22) and (4.10)

$$\frac{\dot{V}}{\alpha} = \frac{(\tilde{\theta}^T \phi)^2}{s^2} - \tilde{\theta}^T M \tilde{\theta} + \frac{\tilde{\theta}^T \tilde{\theta}}{R^2}$$

$$+ 2\tilde{\theta}^T \phi \frac{\nu}{S^2} - 2 \lambda\tilde{\theta}^T \hat{\theta} - \frac{2\tilde{\theta}^T P^{-1}\dot{\theta}^*}{\alpha} + (\frac{\nu^2 - \nu^2}{s^2})$$

Thus

$$\frac{\dot{V}}{\alpha} + \frac{\nu^2}{s^2} = - V + (\frac{\tilde{\theta}^T \tilde{\theta}}{R^2} - 2 \lambda \tilde{\theta}^T \hat{\theta}) + \frac{(\tilde{\theta}^T \phi)^2 + 2 \tilde{\theta}^T \phi\nu + \nu^2}{s^2}$$

$$- \frac{2\tilde{\theta}^T - P^{-1}\dot{\theta}^*}{\alpha} \tag{4.27}$$

consider that $(\tilde{\theta}^T \phi + \nu) = (\hat{\theta} - \theta^*)^T \phi + (z - \phi^T\hat{\theta}) = z - \phi^T\theta^* = v$

thus

$$\frac{\dot{V}}{\alpha} + \frac{\nu^2}{s^2} = - V + (\frac{\tilde{\theta}^T \tilde{\theta}}{R^2} - 2 \lambda \tilde{\theta}^T\hat{\theta}) + \frac{v^2}{s^2} - \frac{2\tilde{\theta} P^{-1} \dot{\theta}^*}{\alpha} \tag{4.28}$$

Now, consider the following terms of the second member of 4.28 :

$$\boxed{\frac{\tilde{\theta}^T \tilde{\theta}}{R^2} - 2\lambda \tilde{\theta}^T \hat{\theta} \leqslant 4} \qquad (4.29)$$

In effect

- if $\|\hat{\theta}\| < R$, then $\lambda = 0$

Then $\|\hat{\theta}\| \leqslant \|\hat{\theta}\| + \|\theta^*\| \leqslant 2R$, which yields (4.29)

- If $|\theta| \geq R$, then $\lambda = 1$ and

$$\frac{\tilde{\theta}^T \tilde{\theta}}{R^2} - 2\lambda \tilde{\theta}^T \hat{\theta} = \frac{1}{R^2} \left[\tilde{\theta}^T \tilde{\theta} - 2\tilde{\theta}^T \hat{\theta} + \hat{\theta}^T \hat{\theta} - \hat{\theta}^T \hat{\theta} \right]$$

$$= \frac{1}{R^2} \theta^{*T} \theta^* - \frac{1}{R^2} \hat{\theta}^T \hat{\theta}$$

$$\leq \frac{1}{R^2} \theta^{*T} \theta^*$$

≤ 1, which yields again (4.29) □

(ii) from hypothesis (4.2) $\boxed{\frac{v^2}{s^2} \leq 4}$ (4.30) □

(iii) $$\boxed{2 \frac{|\tilde{\theta}^T P^{-1} \dot{\theta}^*|}{\alpha} \leq 2\sqrt{V}} \qquad (4.31)$$

In effect, from the Schwartz inequality :

$$|\tilde{\theta}^T P^{-1} \dot{\theta}^*| \leq \sqrt{\tilde{\theta}^T P^{-1} \tilde{\theta}} \sqrt{\dot{\theta}^{*T} P^{-1} \dot{\theta}^*}$$

Then, using (4.26b), (4.24), and (4.6) :

$$|\tilde{\theta}^T P^{-1} \dot{\theta}*| \leq \sqrt{V} \sqrt{\| \dot{\theta}* \|^2 \frac{R^2 + r^2}{R^2 r^2}} \leq \alpha \sqrt{V} \quad \square$$

By substitution of (4.29), (4.20) and (4.31) into (4.28)

$$\frac{\dot{V}}{\alpha} + \frac{\nu^2}{s^2} \leq - V + 4 + 4 + 2\sqrt{V} \tag{4.32}$$

which yield

$$\frac{\dot{V}}{\alpha} \leq - (V - 2\sqrt{V} - 8) = - (\sqrt{V} - 4)(\sqrt{V} + 2) \tag{4.33}$$

Thus $\dot{V}$ becomes negative for $\sqrt{V} > 4$. Knowing that $0 < V(0) \leqslant 16$, it follows that V is always lower than 16.

Comments

(4.25) defines a continuity domain arround $\hat{\theta}$, which is not trivial, In other related works, similar equality arises, such as

$$\hat{\theta} P^{-1} \tilde{\theta} \leqslant V_M$$

Where V_M is deduced from the upper bounds of $\|\theta*\|$, $\|\hat{\theta}\|$, and P^{-1}, so that the above certainty domain be trivial, including the given a priori domain $\|\theta*\| < R^2$!

In our case, V_M is independant of all the other parameters ($V_M = 16$), and the lower bound of P may be very small, if $r^2 \ll R^2$. Thus if P becomes small, the certainty domain may be reduced to a very little subspace of a priori domain.

Property 3 : $\hat{\theta}$ is bounded by

$$\boxed{\|\hat{\theta}\| \leqslant 5\,R} \tag{4.34}$$

In effect, from (4.25), where $P \leqslant R^2 I$, it follows $\|\hat{\theta} - \theta^*\| \leqslant 4\,R$ and from $\|\theta^*\| \leqslant R$, it yields (4.34)

Property 4 : an inequality for ν.

For any positive t and T

$$\boxed{\int_t^{t+T} \frac{\nu^2}{s^2}\, d\tau \leqslant 16 \left(T + \frac{1}{\alpha} \right)} \tag{4.35}$$

Proof

Recall (4.32), in which $\sqrt{V} \leqslant 4$, it yields

$$\frac{\dot{V}}{\alpha} + \frac{\nu^2}{s^2} \leqslant 16$$

Then, by integration :

$$\frac{V(t+T) - V(t)}{\alpha} + \int_t^{t+T} \frac{\nu^2}{s^2}\, d\tau \leq 16T$$

wich yields (4.35), using $0 \leqslant V(t)$ and $V(t+T) \leqslant 16$

Property 5 : an inequality for $\dot{\hat{\theta}}(t)$

For any positive t and T

$$\boxed{\int_t^{t+T} \|\dot{\hat{\theta}}\|\, d\tau \leq 16\sqrt{n}\, R\, \alpha\, (T + 1/\alpha)} \tag{4.36}$$

Proof

From (4.10) $$\|\dot{\hat{\theta}}\| \leq \left\|\frac{\alpha P \phi \nu}{s^2}\right\| + \left\|\frac{\alpha P \hat{\theta}}{R^2}\right\| \tag{4.37}$$

where $\|P\| \leq R^2$, $\|\hat{\theta}\| \leq 5R$

Thus

$$\int_t^{t+T} \|\dot{\hat{\theta}}\| d\tau \leq \int_t^{t+T} \left\|\frac{\alpha P \phi \nu}{s^2}\right\| d\tau + \int_t^{t+T} 5R\alpha \, d\tau \tag{4.38}$$

Consider the term $\left\|\frac{\alpha P \phi \nu}{s^2}\right\| = \left\|\frac{\alpha P \phi}{s}\right\| \left|\frac{\nu}{s}\right|$

Applying the Schwartz inequality

$$\left\{\int_t^{t+T} \left\|\frac{\alpha P \phi}{s}\right\| \left|\frac{\nu}{s}\right| d\tau\right\}^2 \leq \int_t^{t+T} \left\|\frac{\alpha P \phi}{s}\right\|^2 d\tau$$

$$\int_t^{t+T} \left|\frac{\nu}{s}\right|^2 d\tau \tag{4.39}$$

On the other hand

$$\left\|\frac{P \phi}{s}\right\|^2 = \mathrm{Tr}\left(\frac{P \phi \phi^T P}{s^2}\right) = \mathrm{Tr}\left(P - \frac{P^2}{R^2} - \frac{\dot{P}}{\alpha}\right) \quad \text{(from 4.11)}$$

$$\leq \mathrm{Tr}\left(R^2 I - \frac{\dot{P}}{\alpha}\right)$$

Thus

$$\int_t^{t+T} \left\| \frac{P\,\phi}{s} \right\|^2 d\tau \leq Tr\,(R^2 I)\,T + \frac{Tr[P(t)] - Tr[P(t+T)]}{\alpha}$$

$$\leq Tr\,(R^2 I)\,T + \frac{Tr(R^2 I)}{\alpha} = T_r(I) R^2 (T + \frac{1}{\alpha})$$

I beeing the 2n-unit matrix : $Tr(I) = 2n$

Thus

$$\int_t^{t+T} \left\| \frac{P\,\phi}{s} \right\|^2 d\tau \leq 2n\,R^2\,(T + 1/\alpha) \qquad (4.40)$$

Then substitute (4.35) and (4.40) into (4.39)

$$\int_t^{t+T} \left\| \frac{\alpha\,P\,\psi\,v}{s^2} \right\| d\tau^{\;2} \leq 32n\,\alpha^2\,R^2\,(T + \frac{1}{\alpha})^2 \qquad (4.41)$$

Using (4.38) :

$$\int_t^{t+T} \| \dot{\theta} \| \, d\tau \leq \sqrt{32n}\,\alpha\,R\,(T + \frac{1}{\alpha}) + 5\,\alpha\,RT$$

$$\leq (\sqrt{32n} + 5)\,\alpha\,RT + \sqrt{32n}\,R \qquad (4.42)$$

$$\leq (\sqrt{32n} + 5)\,\alpha\,R(T + \frac{1}{\alpha})$$

knowing that

$$\sqrt{32n} + 5 \leq (\sqrt{32} + 5)\sqrt{n} < 16\sqrt{n}, \text{ (because } n \geq 1)$$

thus (4.36) is proved.

5. ROBUST STABILITY OF AN ADAPTIVE CONTROL, ASSUMING THE STABILIZABILITY OF THE IDENTIFIED MODEL

The robust stability results from the following theorem.

THEOREME 2

Let $\mathcal{P}$ a process, parameters $\theta^*(t)$ satisfying (4.1) to (4.6).

Let $\hat{\theta}(t)$ the identified parameters, identified by (4.10) to (4.15).

Assume (as a temporary extra-hypothesis), that $\hat{\theta}(t)$ is admissible, for every t, with respect to the given adjustment law $\mathcal{A}$, the admissible domain beeing defined by (2.18) to (2.21).

Then the closed loop system is uniformly stable if :

$$\alpha > 0 \tag{5.1}$$

$$16 \sqrt{n}\, R\, \alpha \leqslant k_d/M_d \tag{5.2}$$

$$4 \sqrt{2}\, r \leqslant k_x \tag{5.3}$$

$$4 \sqrt{2}\, \rho \leqslant k_\xi \tag{5.4}$$

$$1/\beta \leqslant T_F \tag{5.5}$$

where T_F, k_x, k_ξ, k_d are defined in (3.9) to (3.13).

Proof

Let :

$$x \triangleq \phi, \quad F \triangleq \left[\begin{array}{c|c} \multicolumn{2}{c}{\hat{\theta}^T(t)} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \;\; 0 \end{matrix} & 0 \\ \hline \multicolumn{2}{c}{k^T(\theta)} \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \;\; 0 \end{matrix} \end{array}\right], \quad w \triangleq \left[\begin{array}{c} \nu \\ \hline 0 \\ \hline k_r(\hat{\theta})y_r \\ \hline 0 \end{array}\right]$$

Then :

- x, F and w satisfy (3.1)
- assuming $\|\phi(0)\|$ finite yields (3.2)
- $\hat{\theta}(t)$ being admissible, (2.18) and (2.19) yields (3.3) and (3.4)
- note $\xi \overset{\Delta}{=} \psi$, define $T_\xi = 1/\beta$, then (3.7) and (3.8) are satisfied

From (2.21), $\hat{\theta}$ being admissible :

$$\|\dot{F}\| \leq M_D \|\dot{\hat{\theta}}\| \qquad (5.7)$$

Then, (3.35) becomes

$$\int_t^{t+T} \frac{\|\dot{F}\|}{k_d} d\tau \leq \frac{M_D}{k_D} \int_t^{t+T} \|\dot{\hat{\theta}}\| \, d\tau$$

From (3.36)

$$\int_t^{t+T} \frac{\|\dot{F}\|}{k_d} d\tau \leq \frac{M_D}{k_D} \; 16 \sqrt{n}\, \alpha \, R \, (T + \frac{1}{\alpha})$$

From (5.2), it yields now (3.5) with $T_1 = 1/\alpha$, which is finite from (5.1).

From definitions (5.6), it follows

$$\frac{\| w\|^2}{(M_w + k_x \| x\| + k_\xi \xi)^2} = \frac{\nu^2 + k_r^2 \; y_r^2}{(M_w + k_x \| \phi\| + k_\xi \psi)^2}$$

using (5.3) and (5.4)

$$\frac{\| w\|^2}{(M_w + k_x \| x\| + k_\xi \xi)^2} \le \frac{\nu^2 + k_r^2 \; y_r^2}{(M_w + 4\sqrt{2}\, r\| \psi\| + 4\sqrt{2}\, r \; \psi)^2}$$

$\hat{\theta}$ beeing admissible

$$k_r^2 \le M_r^2 < \infty$$

Define Y_r as the maximum value of $y_r(t)$

$$y_r^2(t) < Y_r^2 < \infty$$

Then define

$$M_w \triangleq \sqrt{2}(4\sigma + M_r Y_r) < \infty$$

It follows

$$\frac{\| w\|^2}{(M_w + k_x\| x\| + k_\xi \xi)^2} \le \frac{M_r^2 \; Y_r^2 + \nu^2}{(\sqrt{2}\, M_r \; Y_r + 4\sqrt{2}(\sigma + r\| \phi\| + \rho \; \psi))^2}$$

$$\le \frac{M_r^2 \; Y_r^2 + \nu^2}{2M_r^2 \; Y_r^2 + 32s^2}$$

$$\le \frac{1}{2} + \frac{\nu^2}{32s^2}$$

Now, by integration, using (4.34) :

$$\int_t^{t+T} \frac{\|w\|^2 \, d\tau}{(M_w + k_x\|x\| + k_\xi \, \xi)^2} \leqslant \frac{T}{2} + \frac{1}{2}\left(T + \frac{1}{\alpha}\right) = T + \frac{1}{2\alpha}$$

Defining $T_2 = 1/2\alpha < \infty$, inequality (3.6) is satisfied.

Now, all the necessary conditions of the robustness theorem are satisfied, thus the stability is proved.

Comments

In (5.2) to (5.5) the first members of the inequalities concern the model of the process $\mathcal{P}$ (noise, model errors, rate of variation), which are related with the design parameters of the identification algorithm. The second members concern the closed loop system characteristics (ω_F, Ω_F), wich are related with the chosen adjustment law $\mathcal{A}$.

To some extent, k_d, k_x, k_ξ are the measure of the intrinsec robustness of $\mathcal{A}$. It follows also that $\mathcal{A}$ must be as continuous as possible, in order to yield $1/M_d$ large.

k_d, k_x, k_ξ, M_d beeing now assumed given, (5.2) to (5.4) impose α, r, ρ to be small. Then (4.2) to (4.6) bound the non linearities, the unmodelled dynamics components in v, and the magnitude of $\dot{\theta}^*$.

Note that α, r, ρ will be generally very small, but T_F being large, slow unmodelled dynamics are permitted by (5.5).

Moreover, σ is not constrained : it must be only finite. It means that large bounded disturbances do not entail the stability.

Moreover, if the process is strictly linear, time invariant, exactly known model order, the adaptive control will be stable, even for r and $\rho = 0$.

It yields that the normalization (by $s = \sigma + r\|\phi\| + \rho\|\psi\|$) and the lower bound ($P \geqslant I \frac{r^2R^2}{r^2+R^2}$) vanish.

Thus, normalization and lower bound of ρ are **not** necessary features in the strictly linear, time-invariant case.

6. SOLVING THE STABILIZABILITY PROBLEM

The above theorem (§5) **locally** solves the stabilizability problem. In effect, if the a priori domain (4.7) is sufficiently small, and is included in the admissible domain, then $\hat{\theta}(t)$ will remain continously admissible.

However, in most cases, the a priori domain is large, and includes non admissible areas.

Then, if $\hat{\theta}(t)$ reach a nonadmissible area, it is necessary to do something, e.g.:

i) to constrain $\hat{\theta}(t)$ into the admissible domain, by means of some amendement of the above identification algorithm. Notice that it could be very difficult if D_A has a complex shape, and if it is not simply connected. Then jumps are to be emphasized, because $\theta^*(0)$ and $\hat{\theta}(0)$ are not necessarily in a same connected area.

ii) an other solution consists in waiting for the emergence of $\hat{\theta}(t)$ from the non admissible domain ($\bar{D}_A$). However if there is no persistingly exciting inputs, $\hat{\theta}(t)$ can stay indefinitely in $\bar{D}_A$.

iii) even if $\hat{\theta}(t)$ is transientely allowed to enter in $\bar{D}_A$, it is at least necessary to freeze $k(\hat{\theta})$, in order to avoid unreasonable values of $k(\hat{\theta})$ and/or $u(t)$. Then, it can be said that **the model from which k is adjusted can be different of the identified model.** This principle was already used in the past (e.g. in "cautious control").

The same principle is applied here, under the following form : the equations of $\hat{\theta}$ beeing unmodified, then a distinct model $\bar{\theta}(t)$ is defined, from $\hat{\theta}(t)$, such that $\bar{\theta}(t)$ be always admissible, and satisfy the desired inequatities.

Define $\bar{\theta}(t)$ as the solution of

$$\dot{\bar{\theta}} = \alpha \, P\phi \, \frac{\bar{\nu}}{s^2} + \sum_{i=0}^{\infty} \delta(t-t_i) \, \Delta_i \qquad (6.1)$$

where :

- $\bar{\theta}(0) = \hat{\theta}(0) \in D_A$
- $\bar{\nu} \triangleq z - \phi^T \bar{\theta}$
- The instant times t_i in (6.1) are those where $\bar{\theta}(t)$ reaches the frontiers of D_A
- $\delta(t-t_i)$ is the Dirac pulse, at time t_i
- $\Delta_i \triangleq \bar{\theta}(t_i^+) - \bar{\theta}(t_i^-)$ is an arbitrary jump, such that

$\bar{\theta}(t_i^+)$ belongs the intersection of the strongly admissible domain D_{SA} (see end of §2), with an ellipsoïd $E(t_i)$ centered over $\hat{\theta}(t_i)$, and defined as :

$$\{\theta \in E(t_i)\} \Longleftrightarrow \{\|\theta - \hat{\theta}(t_i)\|^2_{P^{-1}(t_i)} \leq 16\} \qquad (6.2)$$

where $\hat{\theta}(t)$ and $P(t)$ result from the above identification algorithm (4.10) to (4.15).

The hypotheses on the process $\mathcal{P}$ are the following

- the assumptions of §4 (eq. (4.1) to (4.6))
- moreover, assume that $\theta^*(t)$ is strictly included in D_{SA} for every t (Fig.6.1)

$$\min_{t \geq 0,\ \theta \in D_{FA}} \|\theta^*(t) - \theta\| \triangleq \Delta > 0 \tag{6.3}$$

- assume that δ (see 2.30) satisfies

$$\delta < R \tag{6.4}$$

- and that

$$\theta \in D_A \Rightarrow \|\theta\| \leq R \tag{6.5}$$

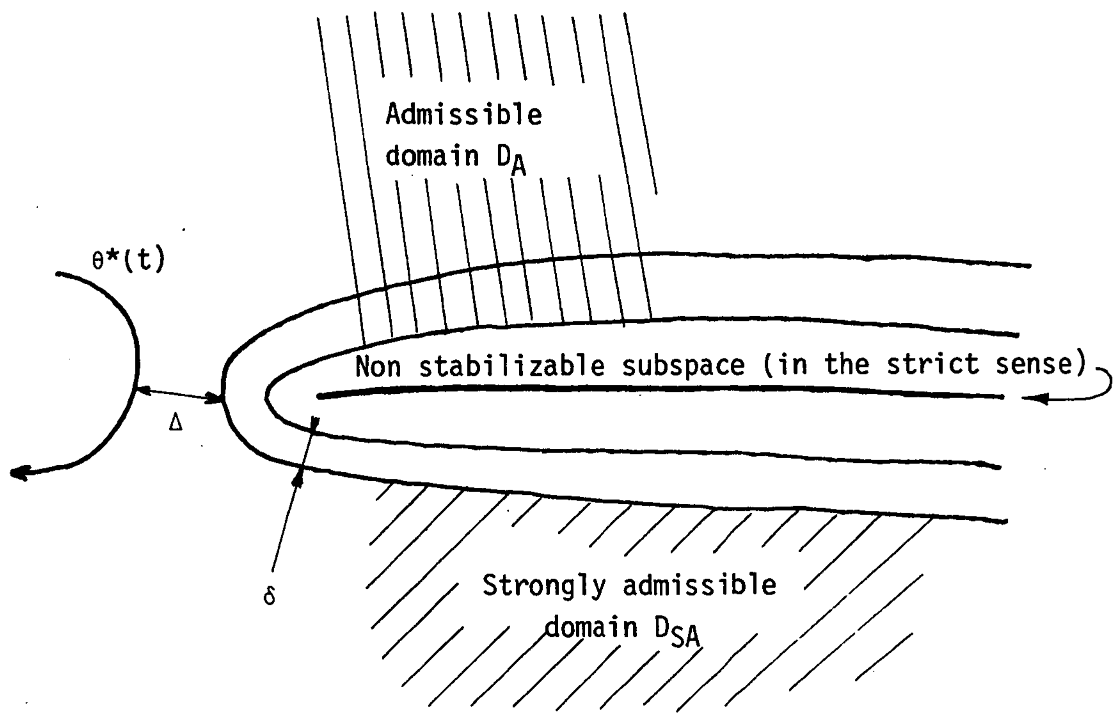

Fig.6.1: The admissible and strongly admissible domains

First, notice that the intersection $D_{FA} \cap E(t_i)$ exists, since it includes at least $\theta^*(t_i)$ (from (6.5), and (4.22) to (4.24)).

Then, a typical trajectory looks like $\overline{\theta}(t)$ on figure 6.2

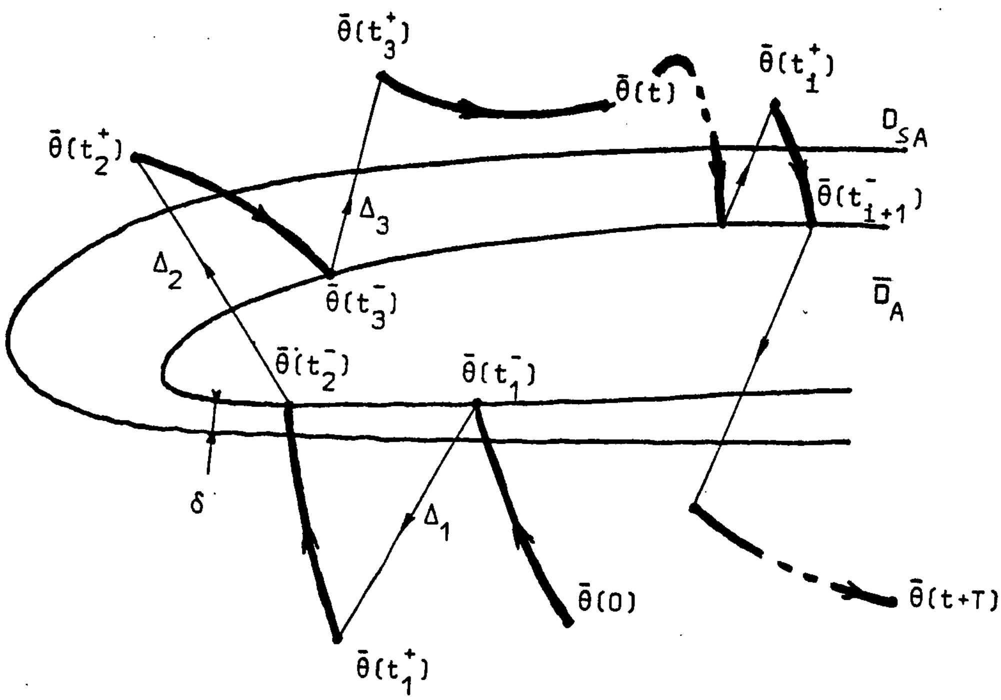

Figure 6.2 : A typical trajectory $\overline{\theta}(t)$

Consider now the search problem for a reinitialization $\bar{\theta}(t_i^+)$.

If it occurs that $\hat{\theta}(t_i) \in D_{FA}$, simply take $\bar{\theta}(t_i^+) = \hat{\theta}(t_i)$.

If $\hat{\theta}(t_i) \not\subset D_{FA}$, recall that :

- $P(t_i)$ is bounded (4.20)
- $\hat{\theta}(t_i)$ is bounded (4.34)
- $\theta^*(t_i)$ is strictly included in $E(t_i)$ - (6.2)
- $D_{FA} \subset D_A$ (bounded, from (6.5))

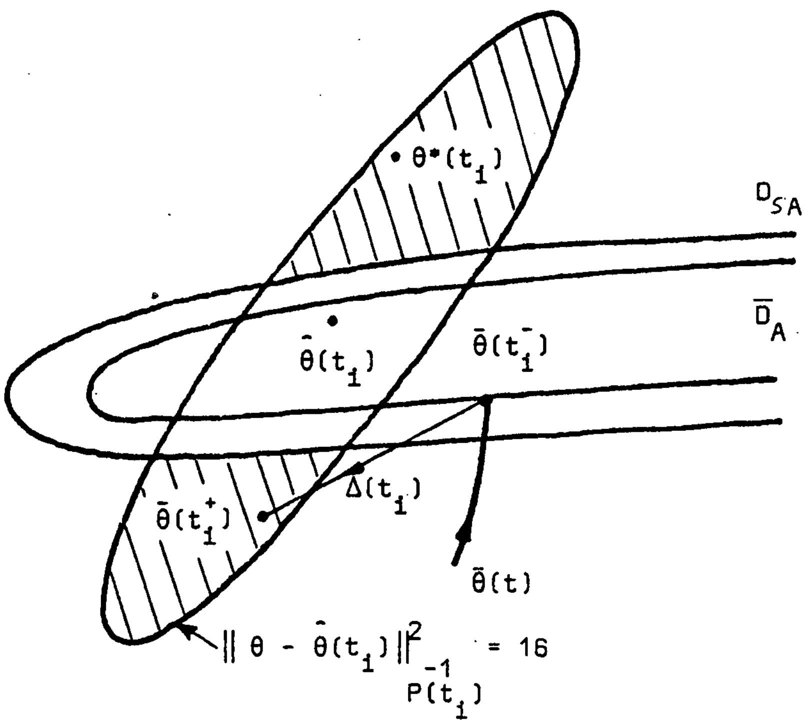

Figure 63 : $\hat{\theta}(t_i) \in D_{SA}$

From examination of the worst case in a figure like (Fig.6.3), it follows that there exists a non zero lower bound for the following ratio :

$$\text{Ratio} = \frac{\text{Volume of } E(t_i) \cap D_{FA} \text{ (dashed)}}{\text{total volume of } E(t_i)} \geq R_m > 0$$

In the limit, R_m approaches unity if $\overline{D}_A$ and $\overline{D}_{FA}$ approach the strictly non stabilizable domain, whose measure is zero (Fig.6.1).

Then, one can design some search algorithm for $\overline{\theta}(t_i^+)$.

In the general case where the shape of D_{FA} is complex and unknown, a simple way lies in implementing a random search, defining

$$\overline{\theta}(t_i^+) = \hat{\theta}(t_i) + P^{1/2}(t_i)\ \gamma \qquad (6.6)$$

where $P^{1/2}$ is a factorization of P, and γ a random vector, uniformly distributed into the radius-4-sphere.

From the existence of R_m, it follows that the expected number of trials will be finite (mean of a Bernouilli variable), and if R_m approaches unity, this number will also approach unity.

Of course, anyone can imagine more sophisticated procedures. However, such procedures must succeed **without any exception** (even if jumps across D_{FA} reveals necessary). In fact, systematic procedures may sometimes decieve due to some unexpected case, so that a random choice can be the most cautious way.

Main properties of the algorithm

a) Consider again the figure (6.2) and define N(t, t+T) as the number of jumps between t an t+T. Define also D(t, t+T) as the length of the trajectory of $\overline{\theta}(t)$, excluding jumps (heavy line on figure 6.2).

From the definition of the distance δ, and the localization of $\overline{\theta}(t_i^-)$ and $\overline{\theta}\ (t_i^+)$ (respectively on the frontier of D_A , and into D_{SA}), it is clear that :

$$\boxed{D(t,\ t+T) \geqslant \left[N(t,t+T) - 1\right]\ \delta} \qquad (6.7)$$

b) define now

$$\bar{V}(t) \triangleq \| \bar{\theta} - \theta^* \|^2_{P^{-1}} \qquad (6.8)$$

At an instant time t_i^+, $\bar{\theta}(t_i^+)$ satisfies

$$\| \bar{\theta}(t_i^+) - \hat{\theta}(t_i) \|^2_{P^{-1}(t_i)} \leq 16$$

and from (4.25)

$$\| \hat{\theta}(t_i) - \theta^*(t_i) \|^2_{P^{-1}(t_i)} \leq 16 \text{ (éq. (4.25))}$$

thus

$$\bar{V}(t_i^+) \leq 4\text{x}16 = 64 \qquad (6.9)$$

Similarly to the proof of (4.24), from t_i^+ as initial time, it follows :

$$\boxed{\bar{V}(t) \leq 64 \qquad \text{for every } t} \qquad (6.10)$$

c) Using (6.10), and similarly with (4.35)

$$\int_{t_i^+}^{t_{i+1}^-} \left(\frac{\bar{\nu}}{s}\right)^2 d\tau \leq 64 \left[t_{i+1} - t_i + \frac{1}{\alpha} \right] \qquad (6.11)$$

Then, if N reinitializations occur between t and t+T

$$\boxed{\int_t^{t+T} \left(\frac{\bar{\nu}}{s}\right)^2 d\tau \leq 64 \left[T + \frac{N+1}{\alpha} \right]} \qquad (6.12)$$

d) From (6.1), D(t, t+T) is given by

$$D(t, t+T) = \int_t^{t+T} \left\| \alpha P \phi \frac{\nu}{s^2} \right\| d\tau$$

Then, from (4.40), (6.12) and the Schwartz inequality

$$D^2(t,\ t+T) \leq \alpha^2 .\ 2n\ R^2(T + \frac{1}{\alpha}) \ .\ 64 \left[T + \frac{N+1}{\alpha} \right] \tag{6.13}$$

Using (6.7) :

$$(N-1)^2\ \delta^2 \ \leq\ 128n\ R^2\ (\alpha T + 1)\ (\alpha T + N + 1)$$

Now, define

$$\boxed{N_0 \ \triangleq\ \frac{128n\ R^2}{\delta^2}} \tag{6.14}$$

It follows

$$(N-1)^2 \ \leq\ N_0(\alpha T + 1)\ (\ N - 1 + \alpha T + 2)$$

Thus

$$(N-1)^2 \ \leq\ N_0(\alpha T + 1)\ (N-1) + N_0(\alpha T + 1)\ (\alpha T + 2) \tag{6.15}$$

For simplicity, the last term is bounded using $N_0 > 1$ (from 6.4) and $\alpha\ T < 2\ \alpha\ T$

$$N_0(\alpha T + 1)(\alpha T + 2) \leq N_0\ \left[N_0(\alpha T + 1)(2\alpha T + 2)\right] = 2\ N_0^{\ 2}(\alpha T + 1)^2$$

(6.15) becomes

$$(N-1)^2 - N_0(\alpha T + 1)(N-1) - 2N_0^{\ 2}(\alpha T + 1) \ \leq\ 0$$

By factorization

$$\left[\ (N-1) + N_0(\alpha T + 1)\ \right]\ \left[\ (N-1) - 2N_0(\alpha T + 1)\ \right] \ \leq\ 0$$

Thus

$$-\ N_0(\alpha T + 1) \leq \underline{(N-1) \leq 2N_0(\alpha T + 1)}\ .$$

and thus

$$\boxed{N(t,\ t + T) \ \leq \ 1 + 2N_0(\alpha T + 1)} \tag{6.16}$$

This key result provides a bound of the possible number of reinitialization between times t and t+T .

Now, it will be easy to bound $\overline{\nu}^2$ and $\overline{\theta}(t)$.

e) Substituting (6.16) into (6.12)

$$\int_t^{t+T} \frac{\overline{\nu}^2}{s^2}\, d\tau \leq 64 \left[T + \frac{1 + 2N_0(\alpha T + 1) + 1}{\alpha} \right]$$

$$= 64 \left[T + \frac{2}{\alpha} + 2\ N_0(T + \frac{1}{\alpha}) \right]$$

$$\leq 64 \left[2T + \frac{2}{\alpha} + 2N_0(T + \frac{1}{\alpha}) \right]$$

Define

$$\boxed{N_1 = N_0 + 1} \tag{6.17}$$

It follows

$$\boxed{\int_t^{t+T} \frac{\overline{\nu}^2}{s^2}\, d\tau \ \leq \ 128\ N_1(T + \frac{1}{\alpha})} \tag{6.18}$$

to be compared with (4.35) : $\ldots \leq 16(T + \frac{1}{\alpha})$

f) Similarly, from (6.16) and (6.13)

$$D^2(t,\ t + T) \leq \alpha^2\ 2n\ R^2(T + \frac{1}{\alpha})\ .\ 128\ N_1(T + \frac{1}{\alpha})$$

From (6.15)

$$128n\ R^2 \ = \ N_0\ \delta^2 \ \leq \ N_1\ \delta^2$$

Thus

$$\boxed{D(t,\ t + T) \ \leq \ 2\ \alpha\delta\ N_1(T + \frac{1}{\alpha})} \tag{6.19}$$

On the other hand, the jumps satisfy $\|\Delta_i\| \leqslant 2R$ (from 6.5). It follows the bound for the whole length (now including jumps).

$$\int_t^{t+T} \|\dot{\bar{\theta}}\| \, d\tau \leqslant D(t,t+T) + 2R\, N(t,t+T) \tag{6.20}$$

Transform using (6.16) and (6.19)

$$\int_t^{t+T} \|\dot{\bar{\theta}}\| \, d\tau \leqslant 2\alpha\, N_1\, \delta(T + \frac{1}{\alpha}) + 2R(1+2N_0(\alpha T+1))$$

For simplicity, introduce $\delta < R$, and $(1 + N_0(1+\alpha T)) < 2N_1(1+\alpha T)$ (from 6.17). It follows

$$\boxed{\int_t^{t+T} \|\dot{\bar{\theta}}\| \, d\tau \leqslant 6N_1\, \alpha\, R(T + \frac{1}{\alpha})} \tag{6.21}$$

to be compared with (4.37).

Now, from the new inequatities (6.18) and (6.2), the robustness theorem applies again, and leads easily to the following theorem :

Theorem 3

Let $\mathcal{P}$ the process. The parameters $\theta^*(t)$ satisfy (4.1) to (4.6) and (6.3) to (6.5), where the domains D_A and D_{SA} are related to some given adjustment law satisfying (3.3), (3.4).

The identified model is given by (6.1).

Then, the closed loop system is globally uniformly stable if :

$$\alpha > 0 \tag{6.22}$$

$$4N_1 \alpha R \leqslant k_d/M_d \tag{6.23}$$

$$16\sqrt{N_1}\, r \leqslant k_x \tag{6.24}$$

$$16\sqrt{N_1}\, \rho \leqslant k_\xi \tag{6.25}$$

$$1/\beta \leqslant T_F \tag{6.26}$$

where

$$N_1 \triangleq \frac{128n\, R^2}{\delta^2} + 1 \tag{6.27}$$

and where T_F, k_x, k_ξ, k_d are defined by (3.9) to (3.19)

Comments

From inequalities (6.23) to (6.25), robustness is clearely reduced, due to the number N_1 (eq. 6.27), in which R/δ is necessarily large.

More precisely, for a given process, σ, α, r, ρ, R, are to be defined from the level of noise, non_linearities, under_modellization, and the rate of time variations, but not from (6.23) to (6.25).

The reason is that the existence of $\overline{\theta}$ depends on the existence of the non empty subspace $D_{SA} \cap E(t_i)$.

Thus, the above data (σ, α, r, ρ, R) must be defined from the best avaliable prior knowledges.

Now, consider that δ is introduced in order to ensure the existence of the upper-bound (6.16) for N(t, t+T), even in the worst imaginable case. It follows that transgressing (6.23) to (6.25) fortunately does not implies unstability.

An other interpretation of the conditions (6.22) to (6.25) is that for a given indirect adaptive control, defined from non zero coefficients α, r, ρ. There exist a non empty class of veakly undermodelized and time varying processes, for which stability is ensured.

7. CONCLUSIONS

The main contributions of the paper are the following :

First, a theorem of robust stability is provided, dealing with weakly undermodelized systems (dynamics and non-linearities). This theorem belongs to the so called "small gain" family. As an original feature, it may deal also with slowly time varying systems.

Moreover, the given upper bounds are possibly satisfied not only by the "exact" models of the processes, but by the classically identified models (typically R.L.S. algorithms).

It follows that the above theorem plays the role of a basic technical Lemma, when analyzing indirect adaptive control systems.

If assuming the identified model be stabilizable for every time, the application of the theorem is rather easy and our results are not really stronger than those of PRALY [14], from which ours are derived.

The main resemblance lies in signals normalization by $1/\|\phi\|$, but there are some differences in the way for bounding $\hat{\theta}$ and P. Moreover, we are dealing with adaptive control of slowly time varying systems.

Then, we proposed, just as a prototype, an identification algorithm satisfying the conditions which are required for applying the robustness theorem. Among other properties, the pair $\hat{\theta}$, P defines a certainty ellipsoid around $\hat{\theta}$. This domain is non-trivial, in the following sense : if P becomes small, the domain really reduce to a small domain (better than the given a priori knowledges).

The existence of this certainty domain already is, by itself,an usefull property. Moreover, it is the basis for producing a new model $\overline{\theta}(t)$, distinct from $\hat{\theta}(t)$, which is not only stabilizable for every t, but also satisfies the required conditions when applying the robustness theorem.

Up to now, the proof concerns only one "prototypical" identification algorithm. However, the main feature of this algorithm are rather common, and generalizations will be provided in the next future.

The only goal of this paper was to prove the **existence** of some fair solution to globally stable adaptive control in the purely passive approach.

Obviously, it remains thrue that exciting imputs are a propitious factor for identification, and then for adaptive control. However it is very important to clain that exciting imputs are **not** a **necessary** condition for robust overall stability of adaptive control.

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APPENDIX

PROOF OF THE ROBUSTNESS THEOREM

First part

Lemma 1

Let Σ be the symmetric solution of

$$F^T\Sigma + \Sigma F + 2I = 0 \qquad \text{A.1}$$

where F satisfies the properties P_F :

$$\left[\begin{array}{ll} \|F\| \leq \Omega < \infty & \text{A.2} \\ R_E(\lambda_i) \leq -\omega_F < 0 & \text{A.3} \end{array}\right.$$

Then :

$$\Sigma \leq T_F I \qquad \text{A.4}$$

$$\Sigma \geq \tau_F I \qquad \text{A.5}$$

Where :

$$T_F \triangleq \frac{n_F^2}{2\omega_F}\left[\frac{2\Omega_F}{\omega_F}\right]^{2n_F} \qquad \text{A.6}$$

$$\tau_F = \frac{1}{\Omega_F} \qquad \text{A.7}$$

Proof

i) For any eigenvector p_i of F :

$$|\lambda_i| = \frac{\| \lambda_i \ p_i \|}{\| p_i \|} = \frac{\| F \ p_i \|}{\| p_i \|} \leq \underset{q \neq 0}{\mathrm{Max}} \frac{\| Fq \|}{\| q \|} \triangleq \| F \|$$

Thus, from (A.2)

$$\boxed{|\lambda_i| \leq \Omega_F} \tag{A.8}$$

ii) Let a(s) be the characteristic polynomial of F :

$$a(s) \triangleq s^{n_F} + a_1 s^{n_F - 1} + \ldots + a_{n_F} = (s - \lambda_1)(s - \lambda_2) \ldots (s - \lambda_{n_F})$$

Then :

$$\left.\begin{aligned}
|a_1| &= \Big| \sum_{i=1}^{n_F} \lambda_i \Big| \leq n_F \ \Omega_F = C^1_{n_F} \ \Omega_F \\
|a_2| &= \Big| \sum_{\substack{i=1 \ j=1 \\ i \neq j}}^{n_F \ \ n_F} \lambda_i \ \lambda_j \Big| \leq C^2_{n_F} \ \Omega_F^{\ 2} \\
&\vdots \\
|a_{n_F - 1}| &\leq C_n^{n-1} \ \Omega_F^{n-1} \\
|a_{n_F}| &\leq \Omega_F^n
\end{aligned}\right\} \tag{A.10}$$

iii) lets consider a factor $\frac{1}{s-\lambda_i}$ of $\frac{1}{a(s)}$

where $\lambda_i = \alpha_i + j\beta_i$, $\alpha_i^2 + \beta_i^2 = |\lambda_i|^2 \leq \Omega_F^{\ 2}$, $\alpha^2 \geq \omega_F^{\ 2}$

then $$\frac{1}{|j\omega - \lambda_i|^2} = \frac{1}{(\omega - \beta)^2 + \alpha^2}$$

and obviously :

$$\boxed{\left| \frac{1}{j\omega - \lambda_i} \right|^2 \leq \frac{1}{\alpha^2} \leq \frac{1}{\omega_F^{\ 2}}} \tag{A.11}$$

there exists also γ satisfying

$$\frac{1}{(\omega-\beta)^2+\alpha^2} \leq \frac{\gamma}{|\lambda_i|^2+\omega^2}$$

γ must satisfy :

$$\gamma\left[(\omega-\beta_i)^2+\alpha_i^2\right]-|\lambda_i|^2-\omega^2 \geq 0$$

which yields

$$\gamma(\omega^2-2\beta_i\omega+\beta_i^2+\alpha_i^2)-|\lambda_i|^2-\omega^2 \geqslant 0$$

$$(\gamma-1)\omega-2\beta_i\gamma\omega+(\gamma-1)\ |\lambda_i|^2 \geqslant 0$$

which is always satisfied for

$$\beta_i\gamma = (\gamma-1)\ |\lambda_i|$$

Thus :

$$\gamma = \frac{|\lambda_i|}{|\lambda_i|-|\beta_i|}$$

Other form $\quad : \gamma = \dfrac{|\lambda_i|\ (|\lambda_i|+|\beta_i|)}{(|\lambda_i|-|\beta_i|)\ (|\lambda_i|+|\beta_i|)} = \dfrac{|\lambda_i|^2+|\lambda_i|\ |\beta_i|}{|\lambda_i|^2-\beta_i^2}$.

$$= \frac{|\lambda_i|^2+|\lambda_i|\ |\beta_i|}{\alpha^2} \leqslant \frac{2|\lambda_i|^2}{\alpha_i^2} \leqslant \frac{2\Omega_F^2}{\omega_F^2}$$

Finally :

$$\boxed{\left|\frac{1}{j\omega-\lambda_i}\right|^2 \leq \frac{2\Omega_F^2}{\alpha^2\ (|\lambda_i|^2+\omega^2)} \leq \frac{2\Omega_F^2}{\omega_F^{\ 2}\ (\omega_F^{\ 2}+\omega^2)}} \qquad \text{A.12}$$

An upper bound of $\left|\dfrac{\omega}{j\omega-\lambda_i}\right|$ can also be easily verified :

$$\boxed{\left|\frac{\omega}{j\omega-\lambda_i}\right|^2 \leq \frac{|\lambda_i|^2}{\alpha^2} \leq \frac{\Omega_F^{\ 2}}{\omega_F^{\ 2}}} \qquad \text{A.13}$$

iv) Prove that

$$\boxed{\|\ (j\omega I-F)^{-1}\| \leq \frac{n_F^{\ 2}}{2}\left(\frac{2\Omega_F}{\omega_F}\right)^{2n_F} \frac{1}{\omega_F^{\ 2}+\omega^2}} \qquad \text{A.14}$$

Proof

First, verify that

$$(sI - F)^{-1} = \frac{1}{a(s)} \{(s^{n_F-1} + a_1 s^{n_F-2} + \dots + a_{n_F-2}\, s + a_{n_F-1})I$$
$$+ (s^{n_F-2} + a_1 s^{n_F-3} + \dots + a_{n_F-2})F$$
$$\vdots$$
$$+ (s + a_1)\, F^{n_F-2}$$
$$+ F^{n_F-1}\} \qquad \text{A.15}$$

(by multiplication of (A.15) by (sI-F) and using the Caley-Hamilton theorem : $F^{n_F} + a_1 F^{n_F-1} + \dots + a_{n_F} I = 0$)

Then, using (A.2), (A.10), (A.15), it follows

$$\|(j\omega I - F)^{-1}\| \leq \left|\frac{1}{a(j\omega)}\right| \times \{|\omega|^{n_F-1} + C^1_{n_F}\, \Omega_F |\omega|^{n_F-2} + \dots + C^{n_F-1}_{n_F}\, \Omega_F^{n_F-1}$$
$$+ \Omega_F |\omega|^{n_F-2} + C^1_{n_F}\, \Omega_F^2 |\omega|^{n_F-3} + \dots + C^{n_F-2}_{n_F}\, \Omega_F^{n_F-1}$$
$$\vdots$$
$$+ \Omega_F^{n_F-2}\, |\omega| + C^1_{n_F}\, \Omega_F^{n_F-1}$$
$$+ \Omega_F^{n_F-1}\} \qquad \text{A.16}$$

consider any term of (A.16) defined as :

$$a_i \triangleq \left|\frac{1}{a(j\omega)}\right| \times \Omega_F^{\,i}\, |\omega|^{n_F-1-i}$$

a_i can be rewritten into

$$a_i = \left[\frac{\Omega_F}{|j\omega - \lambda_1|} \times \dots \times \frac{\Omega_F}{|j\omega - \lambda_i|}\right] \times \left[\frac{|\omega|}{|j\omega - \lambda_{i+1}|} \times \dots \times \frac{|\omega|}{|j\omega - \lambda_{n_F-1}|}\right] \times \left(\frac{1}{|j\omega - \lambda_{n_F}|}\right)$$

Then, from (A.11), (A.12) and (A.13) :

$$a_i \leq \left(\frac{\Omega_F}{\omega_F}\right)^i \left(\frac{\Omega_F}{\omega_F}\right)^{n_F-1-i} \left(\frac{\Omega_F}{\omega_F} \sqrt{\frac{2}{\omega_F^2 + \omega^2}}\right)$$

From which :

$$a_1 \leq \left(\frac{\Omega_F}{\omega_F}\right)^{n_F} \sqrt{\frac{2}{\omega_F^2 + \omega^2}}$$

Thus, (A.16) becomes

$$\|(j\omega I - F)^{-1}\|^2 \leq \left(\frac{\Omega_F}{\omega_F}\right)^{2n_F} \frac{2}{\omega_F^2 + \omega^2} \begin{bmatrix} 1 + C^1_{n_F} + \dots + C^{n_F-1}_{n_F} \\ + 1 + C^1_{n_F} + \dots + C^{n_F-2}_{n_F} \\ \cdot \\ \cdot \\ \cdot \\ + 1 + C^1_{n_F} \\ \\ + 1 \end{bmatrix}^2 \qquad \text{A.17}$$

The last factor is given by

$$\left\{\frac{n_F}{2} \left(1 + C^1_{n_F} + C^2_{n_F} + \dots + C^1_{n_F} + 1\right)\right\}^2 = \left\{\frac{n_F}{2} 2^{n_F}\right\}^2$$

Thus (A.17) becomes (A.14) :

$$\|(j\omega I - F)^{-1}\|^2 \leq \frac{n_F^2}{4} \left(\frac{2\Omega_F}{\omega_F}\right)^{2n_F} \frac{2}{\omega_F^2 + \omega^2}$$

(v) End of proving (A.4) and (A.6)

Recall that if

$$PF + F^T P + Q = 0 \qquad \text{A.18}$$

where F is asymptotically stable and Q is a symetric matrix (positive definite or not), then :

$$P = \int_0^\infty e^{F^T \tau} Q \, e^{F \tau} \, d\tau \qquad \text{A.19}$$

Here, $Q = 2I$, $P = \Sigma$.

On the other hand, the Fourier transform of e^{Ft} is $(j\omega I-F)^{-1}$, then, applying the Parseval inequality :

$$\Sigma = 2\int_0^{\infty} e^{F^T\tau}\, e^{F\tau}\, d\tau = 2\int_0^{\infty} (j\omega I - F)^{-T}(j\omega I - F)^{-1}\ \frac{d\omega}{2\pi}$$

Then

$$\|\Sigma\| \leq 2\int_{-\infty}^{+\infty} \|(j\omega I - F)^{-1}\|^2 \frac{d\omega}{2\pi}$$

Recall that

$$2\int_{-\infty}^{+\infty} \frac{1}{\omega_F^2 + \omega^2}\ \frac{d\omega}{2\pi} = \frac{1}{\omega_F}$$

Thus, using (A.14)

$$\|\Sigma\| \leq \frac{n_F^2}{2\omega_F}\left(\frac{2\Omega_F}{\omega_F}\right)^{2n_F}$$

which proves (A.4) and (A.6)

(vi) Proof of (A.5) and (A.7)

Let $\xi(t)$ be a solution of $\dot{\xi} = F\xi$, and define $E \triangleq \xi^T \xi$

Then $\dot{E} = 2\zeta^T F\zeta$

and $|\dot{E}| \leq 2\|\zeta\|^2\ \|F\| \leq 2\, E\Omega_F$

Thus $\underline{-\,2E\,\Omega_F \leq \dot{E}} \leq 2E\,\Omega_F.$

By integration of the above inequality

$$E(t) \geq E(0)\, e^{-2\Omega_F t} = \|\zeta(0)\|^2\, e^{-2\Omega_F t}$$

A last integration gives

$$\int_0^{\infty} E(t)\, dt \geq \frac{\|\zeta(0)\|^2}{2\Omega_F}$$

On the other hand

$$E(t) = \zeta(0)^T \left[e^{F^T t} e^{Ft} \right] \zeta(0)$$

From (A.18) and (A.19) :

$$\int_0^\infty E(t)\, dt = \zeta(0)^T \frac{\Sigma}{2} \zeta(0)$$

Thus, for any $\xi(0)$:

$$\zeta^T(0)\, \Sigma\, \zeta(0) \leq \frac{\| \zeta(0) \|^2}{\Omega_F}$$

which proves (A.5), (A.7)

Second Part

Lemma 2

Now, F is assumed to be time varying, then, the solution of (A.1) becomes $\Sigma(t)$:

$$F^T(t)\Sigma(t) + \Sigma(t)F(t) + 2I = 0 \qquad \text{A.20}$$

Assume that Ω_F and ω_F are constants in (A.2), (A.3).
Then (A.4), (A.5) are true for all t, and :

$$\boxed{\| \dot{\Sigma}(t) \| \leq 2T_F^2 \| \dot{F} \|} \qquad \text{A.21}$$

Proof

By differenciation of (A.20) :

$$F^T \dot{\Sigma} + \dot{\Sigma} F + (\dot{F}^T \Sigma + \Sigma \dot{F}) = 0$$

Apply (A.18), (A.19) with $P = \dot{\Sigma}$ and $Q = (\dot{F}^T \Sigma + \dot{\Sigma})$

$$\dot{\Sigma} = \int_0^\infty e^{F^T(t)\tau} \left[\dot{F}(t)\ \Sigma(t) + \Sigma(t)\ \dot{F}(t) \right] e^{F^T(t)\tau}\ d\tau$$

Then

$$\|\dot{\Sigma}\| \leq 2\|\dot{F}\|.\|\Sigma\|\ \left\|\int_0^\infty e^{F^T\tau}\ e^{F\tau}\ d\tau\right\|$$

$$\leq 2\|\dot{F}\|.\|\Sigma\|\ \left\|\frac{\Sigma}{2}\right\| \leq \|\dot{F}\|\ T_F^2 \quad \text{(from (A.4))}$$

Third part

Introduce the following positive function :

$$W(t) \triangleq x^T(t)\ \Sigma(t)\ x(t) + T_\xi\ \xi^2(t) \qquad \text{A.22}$$

By differenciation, using (3.1) and (3.8)

$$\dot{W} = x^T\ \dot{\Sigma}x + 2x^T\Sigma(Fx + w) - 2(\xi - \|x\|)\ \xi$$

Using (A.1) :

$$\dot{W} = x^T\ \dot{\Sigma}x + 2x^T\Sigma w - 2x^Tx - 2\xi^2 + 2\|x\|\ \xi$$

$$= x^T\dot{\Sigma}x + 2x^T\Sigma w - x^Tx - \xi^2 - \left[\ \|x\|^2 + \xi^2 - 2\|x\|\ \xi\ \right]$$

where $- x^Tx - \xi^2 \leq -\dfrac{x^T\Sigma x}{T_F} - \dfrac{T_\xi\ \xi^2}{T_F}$ (From (A.4) and (3.9))

use again (A.22) :

$$\boxed{\dot{W} \leq x^T\ \dot{\Sigma}x + |\ 2x^T\Sigma w\ | - \frac{W}{T_F}} \qquad \text{A.23}$$

• consider now the first term $x^T\dot{\Sigma}x$ of (A.23) :

From (A.21), (A.5), (A.22), it follows :

$$x^T\dot{\Sigma}x \leq \|x\|^2 \, T_F^2 \, \|\dot{F}\| \leq \frac{x^T\Sigma x}{\tau_F} \, T_F^2 \, \|\dot{F}\| \leq \frac{T_F^2}{\tau_F} \, W \|\dot{F}\|$$

and, from (3.10)

$$\boxed{x^T\dot{\Sigma}x \leq \frac{1}{2T_F} \, \frac{\|\dot{F}\|}{k_d} \, W} \qquad \text{A.24}$$

• Then consider $x^T\Sigma w$. From the Schwartz inequality :

$$|x^T \Sigma w| \leq \sqrt{x^T \Sigma x} \; \sqrt{w^T\Sigma w}$$

From (A.22) and (A.4) :

$$|x^T \Sigma w| \leq \sqrt{W} \; \sqrt{T_F\|w\|^2} \leq \sqrt{T_F} \; \sqrt{W}\, M\left\|\frac{w}{M}\right\| \qquad \text{A.25}$$

where

$$M(t) \triangleq M_w + k_x\|x(t)\| + k_\xi \, \xi(t) \qquad \text{A.26}$$

From (A.5) :

$$M \leq M_w + k_x \sqrt{\frac{x^T \Sigma x}{\tau_F}} + k_\xi \sqrt{\frac{T_\xi \, \xi^2}{T_\xi}}$$

From (A.22) :

$$M \leq M_w + \left(\frac{k_x}{\sqrt{\tau_F}} + \frac{k_\xi}{\sqrt{T_\xi}}\right) \sqrt{W}$$

Then, (A.25) becomes :

$$|x^T\Sigma w| \leq \left[\sqrt{T_F}\, M_w \sqrt{W} + \left(\sqrt{\frac{T_F}{\tau_F}}\, k_x + \sqrt{\frac{T_F}{T_\xi}}\, k_\xi\right) W \right] \left\|\frac{w}{M}\right\|$$

and from (3.11) and (3.12) :

$$\boxed{2|x^T\Sigma w| \leq \left[2\sqrt{T_F}\, M_w \sqrt{W} + \frac{1}{4T_F} W \right] \left\|\frac{w}{M}\right\|} \qquad \text{A.27}$$

Now, from (A.23), (A.24) and (A.27) :

$$\frac{\dot{W}}{W} \leq -\frac{1}{T_F} + \frac{1}{2T_F}\,\frac{\|\dot{F}\|}{k_d} + \left(\frac{2\sqrt{T_F}\,M_w}{\sqrt{W}} + \frac{1}{4T_F}\right)\left\|\frac{w}{M}\right\| \qquad \text{A.28}$$

Define W_0 as :

$$W_0 \overset{\Delta}{=} \text{Max}\ \{\ W(0)\ ,\ (8T_F\sqrt{T_F}\,M_w)^2\ \}$$

and t_0 a time t (if it exists) such that $W(t_0) = W_0$.
Then, as long as $W(t) \geqslant W_0$, it yields

$$\frac{2\sqrt{T_F}\,M_w}{\sqrt{W}} \leq \frac{1}{4T_F},$$

and A.28 becomes

$$\frac{\dot{W}}{W} \leq -\frac{1}{T_F} + \frac{1}{2T_F}\,\frac{\|\dot{F}\|}{k_d} + \frac{1}{2T_F}\left\|\frac{w}{M}\right\| \qquad \text{A.29}$$

Apply the Schwartz inequality :

$$\left[\int_t^{t+T}\left\|\frac{w}{M}\right\|.\ 1\ d\tau\right]^2 \leq \int_t^{t+T}\left\|\frac{w}{M}\right\|^2 d\tau\ \int_t^{t+T} 1\ .\ d\tau$$

From (3.6) :

$$\left[\int_t^{t+T}\left\|\frac{w}{M}\right\| d\tau\right]^2 \leq (T + T_2)T \qquad (\leq (T + T_2)^2)$$

Thus

$$\int_t^{t+T}\left\|\frac{w}{M}\right\| d\tau \leq T + T_2 \qquad \text{A.30}$$

By integration of (A.29), using (3.5) and (A.30) :

$$\int_{t_0}^{t_0+T}\frac{\dot{W}}{W}\,d\tau \leq + \frac{1}{2T_F} + T_1 + \frac{1}{2T_F} + T_2 = \frac{T_1 + T_2}{2T_F}$$

Thus

$$\operatorname{Log}\left(\frac{W(t_0+t)}{W(t_0)}\right) \leq W_0 \frac{T_1+T_2}{2T_F}$$

and

$$W(t_0+t) \leq W_0 \, e^{\frac{T_1+T_2}{2T_F}}$$

Thus $W(t)$ is uniformly bounded, and from (A.22), (A.5) :

$$\|x\|^2 \leq \frac{x^T \Sigma x}{\tau_F} \leq \frac{W}{\tau_F} \leq \frac{W_0}{\tau_F} \, e^{\frac{T_1+T_2}{2T_F}} < \infty$$

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique